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Exponentially Fitted Spline in Compression for the Numerical Solution of Singular Perturbation Problems

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Abstract—Some exponentially fitted difference schemes for singularly perturbed two point boundary value problems are derived using spline in compression. These schemes are second-order accurate. Numerical examples are given in support of the theoretical results.

Keywords—Singular perturbation, Boundary value problems, ODEs, Spline in compression, Fitting factor.

1. INTRODUCTION

Consider the singularly perturbed problem

$$\begin{gathered} Ly \equiv \varepsilon y'' + a(x)y' + b(x)y = f(x), \\ y(0) = \alpha_0, \qquad y(1) = \alpha_1, \qquad \alpha_0, \alpha_1 \in R, \end{gathered} \tag{1.1}$$

where $a(x)$, $b(x)$, and $f(x)$ are sufficiently smooth with $a(x) > c > 0$, $b(x) > 0$, c is some constant and ε is a small positive parameter.

This class of problems arises in various fields of science and engineering, for instance, fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, geophysics, etc.

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There is a wide variety of asymptotic expansion methods available for solving the problems of the above type. But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions, which are not routine exercises but require skill, insight, and experimentations. In view of the wealth of the literature available on singular perturbation problems and in view of the specialised skills and experience that experts in the field deem necessary, one can raise the question whether there may be other ways to attack these problems, ways that are easy to use and ready for computer implementation, ways that are more accessible to the practicing engineers or applied mathematicians. The spline technique is one such tool to reach these goals in an optimum way.

It is well known that the classical cubic spline collocation method, when applied to (1.1) (with $b(x) \equiv 0$), has an inherent formal cell Reynolds number limitation; i.e., $ha(x_j)/2\varepsilon$ must be less than or equal to 1. For small ε this leads to spurious oscillations or gross inaccuracies in the approximate solution. In order to avoid these difficulties, one introduces exponential functions into the spline basis; e.g., Flaherty and Mathon [1] used polynomial and tension splines, Chin and Krasny [2] used γ-elliptic splines, and Jain and Aziz [3] used adaptive splines. Stynes and O'Riordan [4] used finite element techniques to tackle such problems.

There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there, whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). Present work deals with the second approach, whereas the work based on the first approach has appeared in [5].

In this paper, we have presented a new approach based on spline in compression. We replace the perturbation parameter ε affecting the highest derivative by a fitting factor $\sigma(x, \varepsilon)$. This factor is determined in such a way that the truncation error of the corresponding scheme for the boundary layer function(s), in the case of constant coefficients, should be equal to zero. This procedure is known as the exponential fitting or the introducing of artificial viscosity [6,7]. By making use of the continuity of the first-order derivative of the spline function, the resulting spline difference scheme gives a tridiagonal system which can be solved efficiently by the well-known algorithms. We consider two types of problems. First, we analyse the problems in which the second derivative term and the function term are present while the term containing the first derivative is absent. The problems having the second and first derivative terms but lacking the function term are considered in the second case.

In Section 2, we give a brief description of the method. The derivation of the difference schemes for both of the cases has been given in Section 3. In Section 4, we have determined the fitting factor, whereas in Section 5, the second-order accuracy of the method is shown. In Section 6, we have solved six numerical examples to demonstrate the applicability of the proposed method. The discussion on our results along with some comparisons with the results obtained earlier by others are given in Section 7. The difficulties associated with the most general case are also described in the discussion part.

2. DESCRIPTION OF THE METHOD

For $x \in [x_{j-1}, x_j]$, we define $\tilde{a}(x) = (a_{j-1} + a_j)/2$ and analogously $\tilde{b}(x)$ and $\tilde{f}(x)$ too. Consider first the equation (1.1), with $a(x) \equiv 0$. Therefore, we need to solve

$$\begin{gathered} Ly \equiv \varepsilon y'' + b(x)y = f(x), \\ y(0) = \alpha_0, \qquad y(1) = \alpha_1, \qquad \alpha_0, \alpha_1 \in R. \end{gathered} \tag{2.1}$$

We define the fitting comparison problem associated with (2.1) by

$$\begin{gathered} Ly \equiv \sigma(x, \varepsilon)y'' + b(x)y = f(x), \\ y(0) = \alpha_0, \qquad y(1) = \alpha_1, \qquad \alpha_0, \alpha_1 \in R, \end{gathered} \tag{2.2}$$

where $\sigma(x,\varepsilon)$ is an exponential fitting factor which is to be determined subsequently.

The approximate solution of this problem is sought in the form of the function $S(x)$, which on each interval $[x_{j-1},x_j]$ (denoted by $S_j(x)$) satisfies the following relations:

(i) the differential equation

$$\sigma_j S_j''(x) + \tilde{b}(x)S_j(x) = \tilde{f}(x), \tag{2.3}$$

(ii) the interpolating conditions

$$S_j(x_{j-1}) = u_{j-1}, \qquad S_j(x_j) = u_j, \tag{2.4}$$

(iii) the continuity condition

$$S_j'\left(x_j^+\right) = S_j'\left(x_j^-\right), \tag{2.5}$$

(iv) the consistency condition

$$\frac{p_j}{2} = \tan\frac{p_j}{2}, \qquad p_j = h\sqrt{\frac{b_{j-1}+b_j}{2\sigma_j}}, \tag{2.6}$$

where

$$x \in [x_{j-1},x_j], \qquad x_j = jh, \qquad j = 0(1)n, \qquad h = \frac{1}{n}.$$

Solving equation (2.3) with the help of (2.4), we obtain

$$S_j(x) = \frac{1}{-\sin g_j h}\left[A_j \sin g_j(x_{j-1}-x) + B_j \sin g_j(x-x_j)\right] + \frac{\gamma_j}{\beta_j}, \tag{2.7}$$

where

$$A_j = u_j - \frac{\gamma_j}{\beta_j}, \qquad B_j = u_{j-1} - \frac{\gamma_j}{\beta_j}, \qquad g_j = \sqrt{\frac{\beta_j}{\sigma_j}}, \qquad \beta_j = \frac{b_{j-1}+b_j}{2}, \qquad \gamma_j = \frac{f_{j-1}+f_j}{2}.$$

Equation (2.7) together with (2.6) is known as spline in compression [8]. Using this spline function we will derive the difference scheme in Section 3.

For the second case, we need to solve (1.1) (when $b(x) \equiv 0$); i.e.,

$$\begin{gathered} Ly \equiv \varepsilon y'' + a(x)y' = f(x), \\ y(0) = \alpha_0, \qquad y(1) = \alpha_1, \qquad \alpha_0,\alpha_1 \in R. \end{gathered} \tag{2.8}$$

We seek (as in the first case) the approximate solution of problem (2.8) as a solution of the differential equation

$$\sigma_j S_j''(x) + \tilde{a}(x)S_j'(x) = \tilde{f}(x), \tag{2.9}$$

whereas in this case the parameter p_j used in (2.6) will be given by: $p_j = h(a_{j-1}+a_j)/2\sigma_j$.

Solving (2.9) with the help of (2.4), we obtain

$$\begin{aligned} S_j(x) = \frac{1}{F_j}\left[D_j \exp\left(-\frac{\alpha_j x_{j-1}}{\sigma_j}\right) - E_j \exp\left(-\frac{\alpha_j x_j}{\sigma_j}\right)\right] + \frac{E_j - D_j}{F_j}\exp\left(-\frac{\alpha_j x}{\sigma_j}\right) \\ + \frac{\gamma_j x}{\alpha_j} - \frac{\gamma_j \sigma_j}{\alpha_j^2}, \end{aligned} \tag{2.10}$$

where

$$F_j = \left[\exp\left(-\frac{\alpha_j x_{j-1}}{\sigma_j}\right) - \exp\left(-\frac{\alpha_j x_j}{\sigma_j}\right)\right],$$

$$D_j = u_j - \frac{\gamma_j x_j}{\alpha_j} + \frac{\gamma_j\sigma_j}{\alpha_j^2}, \qquad E_j = u_{j-1} - \frac{\gamma_j x_{j-1}}{\alpha_j} + \frac{\gamma_j\sigma_j}{\alpha_j^2}, \qquad \alpha_j = \frac{a_{j-1}+a_j}{2}.$$

3. DERIVATION OF THE SCHEME

Since $S(x) \in C^2[0,1]$, therefore, we have

$$S'_j(x_j) = S'_{j+1}(x_j). \tag{3.1}$$

CASE I. Using (2.7) and (3.1), we obtain the difference scheme

$$Ru_j = Qf_j, \qquad j = 1(1)n-1, \tag{3.2}$$

where

$$\begin{gathered} Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}, \\ Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}, \\ u_0 = \alpha_0, \qquad u_n = \alpha_1, \end{gathered}$$

$$\begin{gathered} r_j^- = 1 + \frac{p_j^2}{4}, \qquad r_j^+ = 1 + \frac{p_{j+1}^2}{4}, \qquad r_j^c = -4 + r_j^- + r_j^+, \\ q_j^- = \frac{h^2}{4\sigma_j}, \qquad q_j^+ = \frac{h^2}{4\sigma_{j+1}}, \qquad q_j^c = q_j^- + q_j^+, \\ p_j = h\sqrt{\frac{\beta_j}{\sigma_j}}, \qquad \beta_j = \frac{b_{j-1} + b_j}{2}, \end{gathered} \tag{3.3}$$

where σ_j is to be determined.

CASE II. Using (2.10) and (3.1), we obtain the difference scheme given by (3.2), where

$$\begin{gathered} r_j^- = 1 - \frac{p_j}{2}, \qquad r_j^+ = 1 + \frac{p_{j+1}}{2}, \qquad r_j^c = -\left(r_j^- + r_j^+\right), \\ q_j^- = \frac{h^2}{4\sigma_j}, \qquad q_j^+ = \frac{h^2}{4\sigma_{j+1}}, \qquad q_j^c = q_j^- + q_j^+, \\ p_j = h\frac{\alpha_j}{\sigma_j}, \qquad \alpha_j = \frac{a_{j-1} + a_j}{2}, \end{gathered} \tag{3.4}$$

where σ_j is to be determined.

REMARK. If we do not use fitting factor, then instead of (3.3), we obtain

$$\begin{gathered} r_j^- = 1 + \frac{p_j^2}{4}, \qquad r_j^+ = 1 + \frac{p_{j+1}^2}{4}, \qquad r_j^c = -4 + r_j^- + r_j^+, \\ q_j^- = q_j^+ = \frac{h^2}{4\varepsilon}, \qquad q_j^c = \frac{h^2}{2\varepsilon}, \\ p_j = h\sqrt{\frac{\beta_j}{\varepsilon}}, \qquad \beta_j = \frac{b_{j-1} + b_j}{2}, \end{gathered} \tag{3.5}$$

and similarly instead of (3.4), we obtain

$$\begin{gathered} r_j^- = 1 - \frac{p_j}{2}, \qquad r_j^+ = 1 + \frac{p_{j+1}}{2}, \qquad r_j^c = -\left(r_j^- + r_j^+\right), \\ q_j^- = q_j^+ = \frac{h^2}{4\varepsilon}, \qquad q_j^c = \frac{h^2}{2\varepsilon}, \\ p_j = h\frac{\alpha_j}{\varepsilon}, \qquad \alpha_j = \frac{a_{j-1} + a_j}{2}. \end{gathered} \tag{3.6}$$

4. DETERMINATION OF THE FITTING FACTOR

CASE I. To obtain a suitable fitting factor $\sigma(x,\varepsilon)$ for this case, we shall use the following lemma.

LEMMA 4.1. *(See [7].) Let $y(x) \in C^4[0,1]$. Let $b'(0) = b'(1) = 0$. Then the solution of problem (1.1) (with $a(x) \equiv 0$) has the form*

$$y(x) = v(x) + w(x) + g(x),$$

where

$$v(x) = q_0 \exp\left[-x\left\{\frac{b(0)}{(-\varepsilon)}\right\}^{1/2}\right], \qquad w(x) = q_1 \exp\left[-(1-x)\left\{\frac{b(1)}{(-\varepsilon)}\right\}^{1/2}\right],$$

q_0 and q_1 are bounded functions of ε independent of x, and

$$\left|g^{(k)}(x)\right| \le N\left(1 + (-\varepsilon)^{1-k/2}\right), \qquad k = 0(1)4.$$

N is a constant independent of ε.

We require that the truncation error for the boundary layer functions should be equal to zero when $b(x) = b = \text{constant}$.

We take a fitting factor in the following way:

$$\sigma_j = \frac{h^2\beta_j}{4}\,\mu(\rho),$$

where $\mu(\rho)$ $(\rho = \sqrt{b/\varepsilon})$ is to be determined.

From the condition $Rv_j = 0$ for $b(x) = b = \text{constant}$, we have

$$\mu(\rho) = \cot^2\left(\frac{\rho h}{2}\right).$$

The condition $Rw_j = 0$, for $b(x) = b = \text{constant}$, will give the same $\mu(\rho)$. Therefore, we define

$$\mu(\rho) = \cot^2\left(\frac{\rho h}{2}\right), \qquad \text{when } b(x) = b = \text{constant},$$

and

$$\mu(\rho_j) = \cot^2\left(\frac{\rho_j h}{2}\right), \qquad \text{when } b(x) \ne \text{constant}.$$

Hence, the variable fitting factor σ_j is defined as

$$\sigma_j = \frac{h^2\beta_j}{4}\,\mu(\rho_j). \tag{4.1}$$

CASE II. In this case, instead of Lemma 4.1, we use the following lemma to obtain a suitable fitting factor.

LEMMA 4.2. *(See [9].) Let $a(x)$, $f(x) \in C^3(0,1)$. Then*

$$y(x) = v(x) + w(x)$$

where

$$v(x) = \left(-\varepsilon\frac{y'(0)}{a(0)}\right)\exp\left(-\frac{a(0)x}{\varepsilon}\right) \quad \text{and} \quad \left|w^{(k)}(x)\right| \le M\left[1 + \varepsilon^{-k+1}\exp\left(-\frac{\delta x}{\varepsilon}\right)\right],$$

$$k = 0(1)4, \qquad \delta = \frac{a}{4},$$

where $0 < a < a(x)$ for all x and M is a positive constant independent of h and ε.

We require that the truncation error for the boundary layer function should be equal to zero when $a(x) = a =$ constant.

We take a fitting factor in the following way:

$$\sigma_j = \frac{h\alpha_j}{2}\,\mu(\rho),$$

where $\mu(\rho)$ $(\rho = a/\varepsilon)$ is to be determined.

From the condition $Rv_j = 0$ for $a(x) = a =$ constant, we have

$$\mu(\rho) = \coth\left(\frac{\rho h}{2}\right).$$

Therefore, we define

$$\mu(\rho) = \coth\left(\frac{\rho h}{2}\right), \qquad \text{when } a(x) = a = \text{constant},$$

and

$$\mu(\rho_j) = \coth\left(\frac{\rho_j h}{2}\right), \qquad \text{when } a(x) \neq \text{constant}.$$

Hence, the variable fitting factor σ_j is defined as

$$\sigma_j = \frac{h\alpha_j}{2}\,\mu(\rho_j). \tag{4.2}$$

5. PROOF OF THE UNIFORM CONVERGENCE

Throughout the paper, M will denote a positive constant which may take different values in different equations (inequalities) but that are always independent of h and ε.

CASE I. The scheme (3.2),(3.3) can be written in the matrix form

$$Au = F,$$

where A is a matrix of system (3.2), u and F are corresponding vectors.

Now, the local truncation $\tau_j(\phi)$ of scheme (3.2) is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j,$$

where $\phi(x)$ is an arbitrary sufficiently smooth function. Therefore,

$$\begin{aligned} \tau_j(y) &= Ry_j - Q(Ly)_j \\ &= R(y_j - u_j) \\ \Rightarrow R(y_j - u_j) &= \tau_j(y) \\ \Rightarrow \max_j |y_j - u_j| &\leq \left\|A^{-1}\right\| \max_j |\tau_j(y)|. \end{aligned} \tag{5.1}$$

In order to estimate the values $|y_j - u_j|$, we will estimate the truncation error $\tau_j(y)$ and the norm of the matrix A^{-1}.

From (4.1), we see that

$$\begin{aligned} \sigma_j - \varepsilon &= -\frac{h^2\beta_j}{4} + \varepsilon\left[\frac{\left\{h\left(\sqrt{\beta_j/\varepsilon}\right)/2\right\}^2}{\sin^2\left\{h\left(\sqrt{\beta_j/\varepsilon}\right)/2\right\}} - 1\right] \\ &\Rightarrow |\sigma_j - \varepsilon| \leq Mh^2; \end{aligned} \tag{5.2}$$

i.e., σ_j approximates ε with the error $O(h^2)$.

ESTIMATION OF TRUNCATION ERROR AND THE NORM OF A^{-1}. From Lemma 4.1, we have

$$\tau_j(y) = \tau_j(v) + \tau_j(w) + \tau_j(g).$$

We will estimate separately the parts of $\tau_j(y)$.

We will start with $v(x)$,

$$Rv_j = r_j^- v_{j-1} + r_j^c v_j + r_j^+ v_{j+1}.$$

Expanding v_{j-1} and v_{j+1} in terms of v_j, we obtain

$$\begin{aligned} Rv_j &= v_j \left[r_j^- \exp\left(ih\sqrt{\frac{b_0}{\varepsilon}} \right) + r_j^c + r_j^+ \exp\left(-ih\sqrt{\frac{b_0}{\varepsilon}} \right) \right] \\ &= v_j \left[-2 + 2\cosh\left(h\sqrt{\frac{b_0}{\varepsilon}} \right) + \left\{ \frac{h^2\beta_j}{4\sigma_j} \right\} \exp\left(ih\sqrt{\frac{b_0}{\varepsilon}} \right) \right. \\ &\quad \left. + \left\{ \frac{h^2\beta_{j+1}}{4\sigma_{j+1}} \right\} \exp\left(-ih\sqrt{\frac{b_0}{\varepsilon}} \right) + \left(\frac{h^2}{4} \right) \left\{ \left(\frac{\beta_j}{\sigma_j} \right) + \left(\frac{\beta_{j+1}}{\sigma_{j+1}} \right) \right\} \right]. \end{aligned}$$

Expanding exponentials and using $\sigma_j = \varepsilon + O(h^2)$ (from (5.2)), we have

$$Rv_j = \frac{h^2}{\varepsilon}(b_j - b_0)v_j + O\left(\frac{h^4}{\varepsilon}\right) \tag{5.3}$$

and

$$\begin{aligned} Q(Lv)_j &= q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1} \\ &= q_j^- \left(\varepsilon v''_{j-1} + b_{j-1}v_{j-1} \right) + q_j^c \left(\varepsilon v''_j + b_j v_j \right) \\ &\quad + q_j^+ \left(\varepsilon v''_{j+1} + b_{j+1}v_{j+1} \right). \end{aligned}$$

Expanding v_{j-1} and v_{j+1} and their derivatives in terms v_j and its derivatives and using Lemma 4.1, we have

$$Q(Lv)_j = v_j \left[q_j^- (b_{j-1} - b_0) \exp\left(ih\sqrt{\frac{b_0}{\varepsilon}} \right) + q_j^c (b_j - b_0) + q_j^+ (b_{j+1} - b_0) \exp\left(-ih\sqrt{\frac{b_0}{\varepsilon}} \right) \right].$$

Expanding exponentials and using (3.3) and (5.2), we have

$$Q(Lv)_j = \frac{h^2}{\varepsilon}(b_j - b_0)v_j + O\left(\frac{h^4}{\varepsilon}\right). \tag{5.4}$$

From equations (5.3) and (5.4), we have

$$|\tau_j(v)| = |Rv_j - Q(Lv)_j| \le M\,\frac{h^4}{\varepsilon}. \tag{5.5}$$

Similarly,

$$Rw_j = \frac{h^2}{\varepsilon}(b_j - b_1)w_j + O\left(\frac{h^4}{\varepsilon}\right), \tag{5.6}$$

$$Q(Lw)_j = \frac{h^2}{\varepsilon}(b_j - b_1)w_j + O\left(\frac{h^4}{\varepsilon}\right), \tag{5.7}$$

$$\Rightarrow |\tau_j(w)| \le M\,\frac{h^4}{\varepsilon}. \tag{5.8}$$

Now

$$\tau_j(g) = Rg_j - Q(Lg)_j, \tag{5.9}$$

where

$$\begin{aligned} Rg_j &= r_j^- g_{j-1} + r_j^c g_j + r_j^+ g_{j+1} \\ &= \left(r_j^- + r_j^c + r_j^+\right) g_j + \left(r_j^+ - r_j^-\right) h g_j' + \left(r_j^+ + r_j^-\right) \frac{h^2}{2!} g_j'' + \cdots \end{aligned}$$

and

$$\begin{aligned} Q(Lg)_j &= q_j^- \left(\varepsilon g_{j-1}'' + b_{j-1} g_{j-1}\right) + q_j^c \left(\varepsilon g_j'' + b_j g_j\right) + q_j^+ \left(\varepsilon g_{j+1}'' + b_{j+1} g_{j+1}\right) \\ &= \left[q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right] g_j + \left[h\left(-q_j^- b_{j-1} + q_j^+ b_{j+1}\right)\right] g_j' \\ &\quad + \left[q_j^- \left(\varepsilon + \frac{h^2}{2} b_{j-1}\right) + q_j^c \varepsilon + q_j^+ \left(\varepsilon + \frac{h^2}{2} b_{j+1}\right)\right] g_j'' + \cdots . \end{aligned}$$

Therefore, from (5.9), we have

$$\tau_j(g) = T_0 g_j + T_1 g_j' + \text{remainder terms},$$

where

$$\begin{aligned} T_0 &= \left(r_j^- + r_j^c + r_j^+\right) - \left(q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right), \\ T_1 &= \left(r_j^+ - r_j^-\right) h - \left(q_j^+ b_{j+1} - q_j^- b_{j-1}\right) h. \end{aligned}$$

Using (3.3) we see that $T_0 = 0$ and

$$T_1 = \frac{h^3}{4} \left[\frac{\beta_{j+1} - b_{j+1}}{\sigma_{j+1}} - \frac{\beta_j - b_j}{\sigma_j}\right].$$

Therefore, using (5.2), we get $|T_1| \le M\, h^4/\varepsilon$. Hence,

$$\begin{aligned} |T_1 g_j'| &\le \left(M \frac{h^4}{\varepsilon}\right) |g_j'| \\ &\le M \frac{h^4}{\varepsilon}, \qquad \text{using Lemma 4.1,} \\ \Rightarrow |\tau_j(g)| &\le M \frac{h^4}{\varepsilon}. \end{aligned} \tag{5.10}$$

From (5.5), (5.8), and (5.10), we have

$$|\tau_j(y)| \le M \frac{h^4}{\varepsilon}. \tag{5.11}$$

ESTIMATE OF $\|A^{-1}\|$. Since $r_j^c < 0$ and $r_j^\pm > 0$, therefore,

$$\|A^{-1}\| \le \max_j \left|r_j^- + r_j^c + r_j^+\right|^{-1}.$$

Now,

$$\begin{aligned} \left|r_j^- + r_j^c + r_j^+\right| &= \left|r_j^- + \left(-4 + r_j^- + r_j^+\right) + r_j^+\right| \\ &= \left|\frac{h^2}{2}\left(\frac{\beta_j}{\sigma_j} + \frac{\beta_{j+1}}{\sigma_{j+1}}\right)\right| \\ &\ge M_1 \frac{h^2}{\varepsilon}, \qquad \text{using (4.1),} \\ \Rightarrow \left|r_j^- + r_j^c + r_j^+\right|^{-1} &\le M \frac{\varepsilon}{h^2}, \qquad \text{where } M = \frac{1}{M_1}, \\ \Rightarrow \max_j \left|r_j^- + r_j^c + r_j^+\right|^{-1} &\le M \frac{\varepsilon}{h^2} \\ \Rightarrow \|A^{-1}\| &\le M \frac{\varepsilon}{h^2}. \end{aligned} \tag{5.12}$$

Hence, from (5.1), (5.11), and (5.12), we have the following theorem.

THEOREM 5.1. *Let $b(x)$, $f(x) \in C^2[0,1]$, and $b(x) \geq b > 0$, $b'(0) = b'(1) = 0$. Let u_j, $j = 0(1)n$, be the approximate solution of (1.1), when $a(x) \equiv 0$, obtained using (3.2),(3.3). Then, there is a constant M independent of ε and h such that*

$$\max_j |y_j - u_j| \leq Mh^2.$$

CASE II. For the error analysis in this cases we have used the comparison functions method developed by Kellogg and Tsan [10] and Berger *et al.* [9]. By a comparison function we mean a function ϕ such that $L\phi_i > 0$, $-N < i < N$, and $\phi_{\pm N} > 0$, where L is a differential operator and N is a positive integer. These functions are used together with the maximum principle to convert the bounds on truncation error to bounds on discretization error.

This method uses the following two lemmas [9].

LEMMA 5.1. MAXIMUM PRINCIPLE. *Let $\{u_j\}$ be a set of values at the grid points x_j, satisfying $u_0 \leq 0$, $u_n \leq 0$, and $Ru_j \geq 0$, $j = 1(1)n-1$. Then $u_j \leq 0$, $j = 0(1)n$.*

LEMMA 5.2. *If $K_1(h,\varepsilon) \geq 0$ and $K_2(h,\varepsilon) \geq 0$ are such that*

$$R(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \geq R(\pm e_j) = \pm\tau_j(y),$$

for each $j = 1, 2, \ldots, n-1$, then the discrete maximum principle implies that

$$|e_j| \leq K_1(h,\varepsilon)|\phi_j| + K_2(h,\varepsilon)|\psi_j|,$$

where $|e_j| = |u_j - y(x_j)|$, for each j and ϕ and ψ are two comparison functions.

We use two comparison functions (as in [9]): $\phi = -2 + x$ and $\psi = -\exp(-\beta x/\varepsilon)$ (β will be taken to be the smallest of various constants appearing in the proof). Therefore, $\phi_j = -2 + x_j$ and $\psi_j = -[\mu(\beta)]^j$, $j = 0(1)n$, where $\mu(\beta) = [r^-(\beta h/\varepsilon)/r^+(\beta h/\varepsilon)] = \exp(-\beta h/\varepsilon)$.

From (4.2), we see that

$$\sigma_j - \varepsilon = \varepsilon\left[\frac{h\alpha_j}{2\varepsilon}\coth\left(\frac{h\alpha_j}{2\varepsilon}\right) - 1\right].$$

Using $x\coth x = 1 + x^2/3 + O(x^4)$ and the fact that the consistency condition for this case requires $Ch < \varepsilon/\alpha_j$, where C is some positive constant, we get

$$|\sigma_j - \varepsilon| \leq Mh; \tag{5.13}$$

i.e., σ_j approximates ε with the error $O(h)$.

REMARK. Using (5.13), we see that the following inequalities hold for $Ch < \varepsilon/\alpha_j$:

$$R\phi_j \geq M\,\frac{h^2}{\varepsilon}, \qquad R\psi_j \geq M(\mu(\beta))^j\,\frac{h^2}{\varepsilon^2}.$$

Now we estimate the truncation error of the scheme (3.2) using (3.4).

We have

$$\tau_j(y) = T_0 y_j + T_1 y_j' + T_2 y_j'' + \text{remainder terms},$$

where

$$\begin{aligned}
T_0 &= r_j^- + r_j^c + r_j^+,\\
T_1 &= h\left(r_j^+ - r_j^-\right) - \left(q_j^- a_{j-1} + q_j^c a_j + q_j^+ a_{j+1}\right),\\
T_2 &= \frac{h^2}{2}\left(r_j^- + r_j^+\right) - \varepsilon\left(q_j^- + q_j^c + q_j^+\right) + h\left(q_j^- a_{j-1} - q_j^+ a_{j+1}\right).
\end{aligned}$$

Using (3.4) we see that $T_0 = 0$, $T_1 = 0$, and

$$T_2 \doteq \frac{h^2}{2}\left[\left(1 - \frac{h\alpha_j}{2\sigma_j}\right) + \left(1 + \frac{h\alpha_{j+1}}{2\sigma_{j+1}}\right)\right] - \frac{\varepsilon h^2}{2}\left(\frac{1}{\sigma_j} + \frac{1}{\sigma_{j+1}}\right) + \frac{h^3}{4}\left(\frac{a_{j-1}}{\sigma_j} - \frac{a_{j+1}}{\sigma_{j+1}}\right).$$

Using $\sigma_j = \varepsilon + O(h)$ (from (5.13)), we get

$$|T_2| \le M\,\frac{h^3}{\varepsilon}.$$

Therefore, using Lemma 4.2, we have

$$\left|T_2 w_j^{(2)}\right| \le M\,\frac{h^3}{\varepsilon}\left[1 + \frac{1}{\varepsilon}\exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right].$$

Also from Lemma 4.2, we have $v_j'' = (-a(0)/\varepsilon)^2 v_j$, and therefore,

$$|\tau_j(v)| \le \frac{Mh^3}{\varepsilon^2}\exp\left(-\frac{a(0)x_j}{\varepsilon}\right) \Rightarrow |\tau_j(y)| \le M\left[\frac{h^3}{\varepsilon} + \frac{h^3}{\varepsilon^2}\exp\left(-\frac{\delta x_j}{\varepsilon}\right)\right].$$

Choosing $K_1 = h^2$ and $K_2 = h^2/\varepsilon$, we see that Lemma 5.2 is satisfied, and therefore, we have the following theorem.

THEOREM 5.2. *Let $\{u_j\}$, $j = 0(1)n$, be a set of values of the approximate solution to $y(x)$ of (1.1), when $b(x) \equiv 0$, obtained using (3.2) and (3.4). Then there are positive constants β and M (independent of h and ε) such that the following estimate holds:*

$$\max_j |y(x_j) - u_j| \le Mh^2\left[1 + \frac{1}{\varepsilon}\exp\left(-\frac{\beta x_j}{\varepsilon}\right)\right].$$

6. TEST EXAMPLES AND NUMERICAL RESULTS

To illustrate the predicted theory, we solve the following problems.

EXAMPLE 1. (See [11].) Consider $\varepsilon y'' + y = 0$; $y(0) = 0$, $y(1) = 1$, whose exact solution is given by

$$y(x) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}, \qquad \varepsilon \ne (n\pi)^{-2}.$$

EXAMPLE 2. (See [12].) Consider $\varepsilon y'' + (\pi^2/4)y = 0$; $y(0) = 0$, $y(1) = \sin(\pi/2\sqrt{\varepsilon})$, whose exact solution is given by

$$y(x) = \sin\left(\frac{\pi x}{2\sqrt{\varepsilon}}\right).$$

EXAMPLE 3. (See [13].) Consider $\varepsilon y'' + \{3/(1 + x^2/\varepsilon)^2\}y = 0$; $y(0) = 0$, $y(0.1) = 0.1/\sqrt{\varepsilon + 0.01}$, whose exact solution is given by

$$y(x) = \frac{x}{\sqrt{\varepsilon + x^2}}.$$

EXAMPLE 4. (See [14].) Consider $\varepsilon y'' + y' = 2$; $y(0) = 0$, $y(1) = 1$, whose exact solution is given by

$$y(x) = 2x + \frac{1 - e^{-(x/\varepsilon)}}{e^{-(1/\varepsilon)} - 1}.$$

EXAMPLE 5. (See [15].) Consider $\varepsilon y'' + (x+1)^3 y' = f(x)$; $y(0) = 2$, $y(1) = (1/8)\exp(-15/4\varepsilon) + \exp(-1/2)$, whose exact solution is given by

$$y(x) = \frac{1}{(x+1)^3}\exp\left[-\frac{1}{4\varepsilon}\left\{(x+1)^4 - 1\right\}\right] + \exp\left(-\frac{x}{2}\right).$$

EXAMPLE 6. (See [16].) Consider $\varepsilon y'' + [2\varepsilon/(1+x) + 2/(1+x)^2]y' = 0$; $y(0) = 0$, $y(1) = 0$, whose exact solution is given by

$$y(x) = \cos\left(\frac{\pi x}{1+x}\right) + \frac{\exp(-1/\varepsilon) - \exp(-2x/(\varepsilon(1+x)))}{1 - \exp(-1/\varepsilon)}.$$

All the tables except Table 11 contain the maximum errors at all the mesh points

$$\max_j |y(x_j) - u_j|$$

for different n and ε, $n = 1/h$.

Table 11 contains the numerical rate of uniform convergence which is determined as in [7],

$$r_{k,\varepsilon} = \log_2\left(\frac{z_{k,\varepsilon}}{z_{k+1,\varepsilon}}\right), \qquad k = 0, 1, \ldots,$$

where

$$z_{k,\varepsilon} = \max_j \left| u_j^{h/2^k} - u_{2j}^{h/2^{k+1}} \right|, \qquad k = 0, 1, \ldots,$$

and $u_j^{h/2^k}$ denotes the value of u_j for the mesh length $h/2^k$.

Table 1. Numerical results for Example 1 (max. error) without using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/2	0.26E−03	0.65E−04	0.16E−04	0.41E−05	0.10E−05	0.26E−06	0.64E−07
1/4	0.19E−02	0.47E−03	0.12E−03	0.29E−04	0.74E−05	0.18E−05	0.46E−06
1/8	0.73E−01	0.19E−01	0.47E−02	0.12E−02	0.29E−03	0.73E−04	0.18E−04
1/16	0.29E−01	0.71E−02	0.18E−02	0.44E−03	0.11E−03	0.28E−04	0.69E−05
1/32	0.14E+00	0.38E−01	0.98E−02	0.25E−02	0.62E−03	0.15E−03	0.39E−04
1/64	0.13E+00	0.34E−01	0.87E−02	0.22E−02	0.55E−03	0.14E−03	0.34E−04
1/128	0.39E+00	0.11E+00	0.28E−01	0.72E−02	0.18E−02	0.45E−03	0.11E−03
1/256	0.46E+01	0.38E+02	0.14E+01	0.27E+00	0.63E−01	0.16E−01	0.39E−02
1/512	0.29E+01	0.58E+01	0.95E+00	0.18E+00	0.42E−01	0.10E−01	0.25E−02

Table 2. Numerical results for Example 1 (max. error) using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/2	0.87E−14	0.61E−13	0.14E−12	0.30E−12	0.98E−12	0.14E−10	0.54E−10
1/4	0.20E−14	0.16E−13	0.11E−12	0.12E−11	0.47E−12	0.21E−10	0.98E−10
1/8	0.89E−13	0.57E−12	0.45E−11	0.93E−11	0.22E−10	0.67E−10	0.96E−09
1/16	0.13E−13	0.10E−13	0.61E−13	0.38E−12	0.47E−11	0.16E−11	0.76E−10
1/32	0.64E−14	0.32E−13	0.32E−12	0.23E−11	0.50E−11	0.12E−10	0.36E−10
1/64	0.65E−14	0.16E−13	0.88E−14	0.69E−13	0.48E−12	0.58E−11	0.15E−11
1/128	0.21E−14	0.42E−14	0.21E−13	0.20E−12	0.17E−11	0.39E−11	0.84E−11
1/256	0.10E−13	0.21E−12	0.39E−12	0.23E−12	0.23E−11	0.13E−10	0.17E−09
1/512	0.18E−13	0.63E−14	0.25E−13	0.13E−12	0.11E−11	0.94E−11	0.22E−10

7. DISCUSSION

We have described a numerical method for solving singular perturbation problems using spline in compression. It is a practical method and can easily be implemented on a computer to solve

Table 3. Numerical results for Example 2 (max. error) without using fitting factor.

$\varepsilon = 2^{-k}$	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
$k = 1$	0.32E−02	0.81E−03	0.20E−03	0.51E−04	0.13E−04	0.32E−05	0.79E−06
$k = 3$	0.20E−01	0.51E−02	0.13E−02	0.32E−03	0.79E−04	0.20E−04	0.49E−05
$k = 5$	0.29E+00	0.96E−01	0.26E−01	0.66E−02	0.17E−02	0.42E−03	0.10E−03
$k = 7$	0.17E+01	0.39E+00	0.11E+00	0.30E−01	0.75E−02	0.19E−02	0.47E−03
$k = 9$	0.18E+01	0.17E+01	0.68E+01	0.32E+00	0.69E−01	0.17E−01	0.42E−02
$k = 11$	0.20E+01	0.52E+01	0.18E+01	0.44E+01	0.43E+00	0.12E+00	0.30E−01
$k = 13$	0.12E+01	0.17E+01	0.27E+01	0.17E+01	0.26E+01	0.80E+01	0.41E+00
$k = 15$	0.33E+01	0.20E+01	0.24E+01	0.23E+01	0.20E+01	0.23E+01	0.59E+01

Table 4. Numerical results for Example 2 (max. error) using fitting factor.

$\varepsilon = 2^{-k}$	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
$k = 1$	0.16E−13	0.77E−13	0.21E−12	0.37E−13	0.11E−11	0.15E−10	0.34E−11
$k = 3$	0.59E−14	0.24E−13	0.11E−12	0.35E−12	0.31E−13	0.17E−11	0.23E−10
$k = 5$	0.14E−14	0.26E−13	0.12E−12	0.60E−12	0.17E−11	0.99E−13	0.98E−11
$k = 7$	0.34E−14	0.18E−14	0.27E−13	0.14E−12	0.71E−12	0.18E−11	0.27E−12
$k = 9$	0.46E−14	0.99E−14	0.41E−14	0.60E−13	0.29E−12	0.16E−11	0.41E−11
$k = 11$	0.64E−14	0.13E−13	0.17E−13	0.11E−13	0.11E−12	0.54E−12	0.29E−11
$k = 13$	0.20E−13	0.19E−13	0.23E−13	0.60E−13	0.20E−13	0.28E−12	0.14E−11
$k = 15$	0.40E−13	0.41E−13	0.33E−13	0.51E−13	0.81E−13	0.34E−13	0.42E−12

Table 5. Numerical results for Example 3 (max. error) without using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/1	0.17E−06	0.44E−07	0.11E−07	0.28E−08	0.69E−09	0.17E−09	0.43E−10
1/2	0.95E−06	0.24E−06	0.61E−07	0.15E−07	0.38E−08	0.95E−09	0.24E−09
1/4	0.51E−05	0.13E−05	0.33E−06	0.82E−07	0.21E−07	0.51E−08	0.13E−08
1/8	0.27E−04	0.67E−05	0.17E−05	0.42E−06	0.11E−06	0.27E−07	0.66E−08
1/16	0.13E−03	0.32E−04	0.81E−05	0.20E−05	0.51E−06	0.13E−06	0.32E−07
1/32	0.53E−03	0.13E−03	0.34E−04	0.84E−05	0.21E−05	0.53E−06	0.13E−06
1/64	0.18E−02	0.44E−03	0.11E−03	0.28E−04	0.71E−05	0.18E−05	0.44E−06
1/128	0.42E−02	0.12E−02	0.29E−03	0.73E−04	0.18E−04	0.46E−05	0.11E−05

Table 6. Numerical results for Example 3 (max. error) using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/1	0.11E−06	0.29E−07	0.73E−08	0.18E−08	0.46E−09	0.11E−09	0.29E−10
1/2	0.63E−06	0.16E−06	0.40E−07	0.10E−07	0.25E−08	0.63E−09	0.16E−09
1/4	0.34E−05	0.85E−06	0.22E−06	0.54E−07	0.13E−07	0.34E−08	0.84E−09
1/8	0.17E−04	0.43E−05	0.11E−05	0.27E−06	0.68E−07	0.17E−07	0.43E−08
1/16	0.80E−04	0.20E−04	0.50E−05	0.13E−05	0.31E−06	0.78E−07	0.20E−07
1/32	0.31E−03	0.77E−04	0.19E−04	0.48E−05	0.12E−05	0.30E−06	0.75E−07
1/64	0.82E−03	0.22E−03	0.54E−04	0.13E−04	0.34E−05	0.84E−06	0.21E−06
1/128	0.15E−02	0.36E−03	0.88E−04	0.22E−04	0.55E−05	0.14E−05	0.34E−06

Table 7. Numerical results for Example 4 (max. error) without using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/2	0.29E−03	0.72E−04	0.18E−04	0.45E−05	0.11E−05	0.28E−06	0.70E−07
1/4	0.17E−02	0.43E−03	0.11E−03	0.27E−04	0.67E−05	0.17E−05	0.42E−06
1/8	0.78E−02	0.19E−02	0.48E−03	0.12E−03	0.30E−04	0.75E−05	0.19E−05
1/16	0.35E−01	0.79E−02	0.19E−02	0.48E−03	0.12E−03	0.30E−04	0.75E−05
1/32	0.14E+00	0.35E−01	0.79E−02	0.19E−02	0.48E−03	0.12E−03	0.30E−04
1/64	0.35E+00	0.14E+00	0.35E−01	0.79E−02	0.19E−02	0.48E−03	0.12E−03
1/128	0.60E+00	0.35E+00	0.14E+00	0.35E−01	0.79E−02	0.19E−02	0.48E−03
1/256	0.81E+00	0.60E+00	0.35E+00	0.14E+00	0.35E−01	0.79E−02	0.19E−02
1/512	0.12E+01	0.78E+00	0.60E+00	0.35E+00	0.14E+00	0.35E−01	0.79E−02

Table 8. Numerical results for Example 4 (max. error) using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/2	0.62E−15	0.13E−13	0.18E−13	0.12E−12	0.21E−12	0.20E−11	0.49E−11
1/4	0.75E−15	0.14E−14	0.22E−13	0.31E−13	0.19E−12	0.68E−12	0.30E−11
1/8	0.39E−15	0.29E−14	0.94E−15	0.30E−13	0.33E−13	0.25E−12	0.22E−11
1/16	0.28E−15	0.67E−15	0.51E−14	0.20E−14	0.79E−13	0.54E−13	0.20E−12
1/32	0.39E−15	0.50E−15	0.89E−15	0.10E−13	0.26E−14	0.20E−12	0.21E−12
1/64	0.42E−15	0.80E−15	0.22E−15	0.58E−14	0.12E−13	0.10E−12	0.77E−12
1/128	0.11E−15	0.11E−14	0.22E−14	0.17E−15	0.11E−13	0.23E−13	0.21E−12
1/256	0.00E+00	0.00E+00	0.15E−14	0.41E−14	0.10E−14	0.23E−13	0.48E−13
1/512	0.00E+00	0.00E+00	0.00E+00	0.42E−14	0.80E−14	0.84E−14	0.50E−13

Table 9. Numerical results for Example 5 (max. error) without using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/2	0.45E−02	0.11E−02	0.28E−03	0.70E−04	0.18E−04	0.44E−05	0.11E−05
1/4	0.63E−02	0.16E−02	0.40E−03	0.99E−04	0.25E−04	0.62E−05	0.15E−05
1/8	0.13E−01	0.32E−02	0.80E−03	0.20E−03	0.50E−04	0.12E−04	0.31E−05
1/16	0.42E−01	0.94E−02	0.23E−02	0.57E−03	0.14E−03	0.36E−04	0.89E−05
1/32	0.14E+00	0.37E−01	0.83E−02	0.20E−02	0.51E−03	0.13E−03	0.32E−04
1/64	0.33E+00	0.14E+00	0.35E−01	0.80E−02	0.20E−02	0.49E−03	0.12E−03
1/128	0.54E+00	0.34E+00	0.13E+00	0.35E−01	0.79E−02	0.19E−02	0.48E−03
1/256	0.72E+00	0.56E+00	0.34E+00	0.14E+00	0.35E−01	0.79E−02	0.19E−02
1/512	0.86E+00	0.73E+00	0.58E+00	0.35E+00	0.14E+00	0.35E−01	0.79E−02

Table 10. Numerical results for Example 5 (max. error) using fitting factor.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
1/2	0.26E−02	0.65E−03	0.16E−03	0.40E−04	0.10E−04	0.25E−05	0.63E−06
1/4	0.13E−02	0.31E−03	0.78E−04	0.19E−04	0.49E−05	0.12E−05	0.30E−06
1/8	0.21E−02	0.62E−03	0.16E−03	0.41E−04	0.10E−04	0.26E−05	0.64E−06
1/16	0.64E−02	0.20E−02	0.54E−03	0.14E−03	0.35E−04	0.86E−05	0.22E−05
1/32	0.78E−02	0.45E−02	0.13E−02	0.34E−03	0.87E−04	0.22E−04	0.54E−05
1/64	0.40E−02	0.56E−02	0.26E−02	0.74E−03	0.19E−03	0.48E−04	0.12E−04
1/128	0.45E−02	0.27E−02	0.32E−02	0.14E−02	0.39E−03	0.10E−03	0.25E−04
1/256	0.47E−02	0.25E−02	0.16E−02	0.17E−02	0.73E−03	0.20E−03	0.52E−04
1/512	0.49E−02	0.26E−02	0.13E−02	0.89E−03	0.89E−03	0.37E−03	0.10E−03

Table 11. Numerical results for Example 5 (rate of convergence) $n = 128, 256, 512, 1024, 2048$.

ε	$r(0)$	$r(1)$	$r(2)$	$r(3)$	$r(4)$	Avg.
1/2	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/4	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/8	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/16	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/32	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/64	0.19E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
1/128	0.18E+01	0.19E+01	0.20E+01	0.20E+01	0.20E+01	0.19E+01
1/256	——	0.18E+01	0.19E+01	0.20E+01	0.20E+01	0.19E+01

Table 12. Numerical Results for Example 6 (max. error) $\varepsilon = 10^{-5}$.

n	$M1$	$M2$	Our Results
20	0.11E−01	0.11E−01	0.17E−01
40	0.62E−02	0.73E−02	0.92E−02
80	0.36E−02	0.43E−02	0.48E−02
160	0.20E−02	0.23E−02	0.24E−02
320	0.11E−02	0.12E−02	0.12E−02
640	0.26E−02	0.62E−03	0.61E−03
1280	0.61E−02	0.38E−03	0.31E−03

$M1$: van Veldhuizen [17]. Upstream difference scheme.

$M2$: van Veldhuizen [17]. Modified upwind difference scheme.

Table 13. Numerical results for Example 6 (max. error) $\varepsilon = 10^{-4}$.

	$n = 10$	$n = 20$	$n = 40$	$n = 80$
van Veldhuizen [16]	0.16E+00	0.79E−01	0.39E−01	0.20E−01
Our Results	0.31E−01	0.17E−01	0.92E−02	0.48E−02

such problems. The method has been analysed for convergence. Test examples have been solved to demonstrate the efficiency of the proposed method.

Results are tabulated for both the spline in compression without fitting factor and with fitting factor and it can be seen from the respective tables that the use of fitting factor is quite advantageous.

For the numerical solution of Example 3, we took $h = 0.1/n$ as the concerned interval for this particular example is $[0, 0.1]$.

Example 6 has been solved earlier by van Veldhuizen [16,17] using finite element methods with positive type higher-order schemes in [17] and piecewise polynomials of degree $\leq k$ (k being any positive integer) in [16]. We obtain quite convincing results with those in [17] and better than those in [16] (for $k = 1$).

The problems of the type as considered in the second case have earlier been solved by Stojanovic [18,19] using exponentially fitted quadratic spline difference schemes, Surla and Jerkovic [20] using exponentially fitted cubic spline collocation method, and by Surla and Uzelac [21] using cubic spline difference schemes, but they could get only first order of uniform convergence, whereas the present method has second order of uniform convergence. Example 5 has also been solved earlier by Sakai and Usmani [15]. Their method works for smaller values of ε, but they too

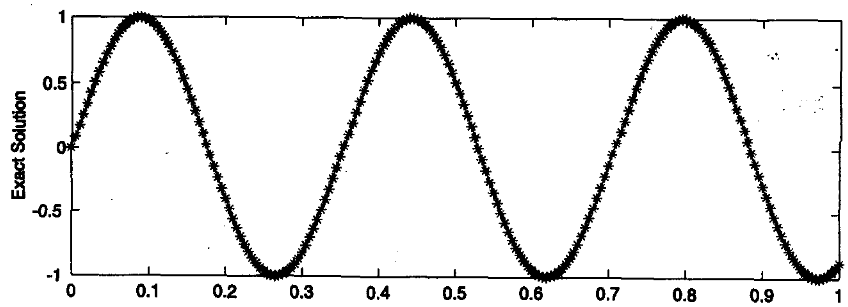

Figure 1. Exact solution for Example 2 for $\varepsilon = 1/128$, $h = 1/200$.

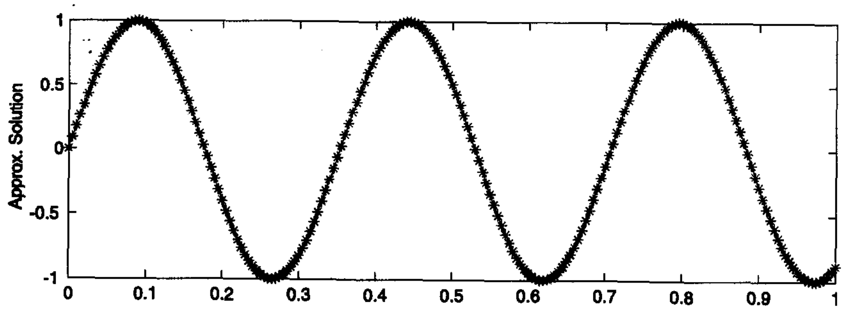

Figure 2. Approximate solution for Example 2 for $\varepsilon = 1/128$, $h = 1/200$.

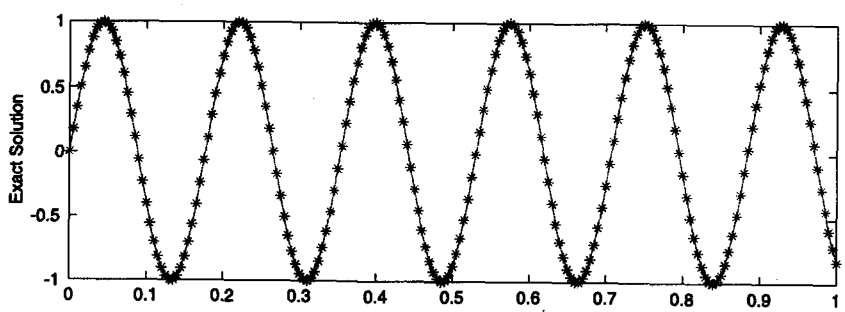

Figure 3. Exact solution for Example 2 for $\varepsilon = 1/512$, $h = 1/200$.

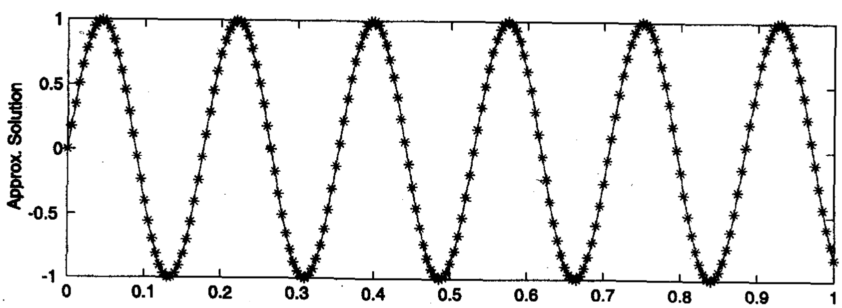

Figure 4. Approximate solution for Example 2 for $\varepsilon = 1/512$, $h = 1/200$.

did not achieve uniform convergence. However, our method works for smaller values of ε also, but in that case we require h also to be very small so as to satisfy the consistency condition.

The solution of the equations considered in the first case is having oscillatory behaviour [1,11]. Also r_js and q_js in the schemes for Case II, without using the consistency condition, involve $\exp(-\kappa(h,\sigma_j))$ and $\exp(+\kappa(h,\sigma_j))$ terms, where $\kappa(h,\sigma_j)$ is a function of h and ε. Therefore, in this case the r_js and q_js will tend to 0 and (or) ∞. Hence, the system thus obtained may not be well behaved. To overcome spurious oscillations in the solution of the equation considered in the first case as well as the above-mentioned difficulties in the second case, we use the consistency condition.

It is obvious from Example 2 that its exact solution is rapidly oscillatory for small ε. To see the behaviour of our computed solutions with this exact solution, graphs have been plotted for values of $x \in [0,1]$ versus these two solutions. We took $n = 200$ and $\varepsilon = 1/128$ for Figures 1 and 2 and $\varepsilon = 1/128$ for Figures 3 and 4. It can be seen that the exact and computed solutions are perfectly identical which further corroborates the applicability of the proposed method.

Finally, we would like to remark that if we consider the most general case, i.e., problem (1.1) and if we replace ε by the fitting factor σ_j, then the scheme thus obtained will be given by

$$r_j^- = \left(1 - \frac{t_j^2}{4}\right)\left(\frac{2-p_j}{2+p_j}\right), \qquad r_j^+ = \left(1 - \frac{t_{j+1}^2}{4}\right)\left(\frac{2+p_{j+1}}{2-p_{j+1}}\right),$$

$$r_j^c = -2 + p_j - p_{j-1} - \frac{1}{4}\left(t_j^2 + t_{j+1}^2\right), \qquad q_j^- = \frac{h^2}{2\sigma_j(2+p_j)}, \qquad q_j^+ = \frac{h^2}{2\sigma_{j+1}(2-p_{j+1})},$$

$$q_j^c = q_j^- + q_j^+,$$

$$p_j = \frac{h\alpha_j}{2\sigma_j}, \qquad t_j = \frac{\left[h\left(\alpha_j^2 - 4\beta_j\sigma_j\right)^{1/2}\right]}{2\sigma_j}.$$

Because of these two parameters p_j and t_j, the factor $\mu(\rho)$, as in the first two cases, cannot be obtained explicitly, and thus, the fitting factor has not been determined for this case.

The computations reported in this paper were done on Silicon Graphics Origin 200 (dual processor) Operating System (in Fortran 77 in double precision with 16 significant figures) at IIT Kanpur.

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